Knox Grammar School

2021

Trial Higher School Certificate Examination

Name: ____________________

Teacher: ____________________

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- The official NESA Reference Sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.

Section I ~ Pages 3-6

- 10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II ~ Pages 7-15

- 90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

Teachers:

Mr Bradford (Examiner)
Ms Yun
Mr Vuletich

Write your name, your Teacher's Name and your Student Number on the front cover of each answer booklet

Number of Students in Course: 31

This examination is based on the CSSA 2021 Extension 2 paper and modified in accordance with the CSSA directives:
Examinations can be compressed by removing some questions to shorten the examination. Schools should only upload/scan the relevant question. The CSSA Trial HSC Examination Timetable is to be adhered to; this includes the security period.

Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1. Consider the statement: "If an animal is a bird, then it can fly or swim."

What is the contrapositive statement?

(A) If an animal cannot fly or cannot swim, then it is a bird.

(B) If an animal cannot fly or cannot swim, then it is not a bird.

(C) If an animal cannot fly and cannot swim, then it is a bird.

(D) If an animal cannot fly and cannot swim, then it is not a bird.

2. Recall that ${}^nC_r$ is defined for non-negative integers r and n where $0 \le r \le n$.

Consider the equation $\dfrac{100\ {}^xC_{100}}{{}^{x-1}C_{99}} = \dfrac{1}{x^{99}}$. How many real solutions in x does this equation possess?

(A) 0

(B) 1

(C) 2

(D) infinite

3. What is the angle between the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, to the nearest degree?

(A) 77°

(B) 83°

(C) 84°

(D) 96°

4. $A(1,2,2)$, $B(3,-12,4)$, $C(1,2,0)$ and $D(3,-12,0)$ are four position vectors.

What is the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$?

(A) $2\underset{\sim}{i} - 14\underset{\sim}{j} + 2\underset{\sim}{k}$

(B) $2\underset{\sim}{i} - 14\underset{\sim}{j} + 4\underset{\sim}{k}$

(C) $2\underset{\sim}{i} - 14\underset{\sim}{j}$

(D) $-2\underset{\sim}{i} + 14\underset{\sim}{j}$

5. For all non-zero integers x and y, if $x > y$ then $\frac{1}{x} < \frac{1}{y}$.

What is a counter-example to the above statement?

(A) $x = 2, y = -1$

(B) $x = 0, y = 0$

(C) $x = 4, y = 3$

(D) $x = -2, y = 1$

6. A particle is moving in simple harmonic motion with displacement x metres. Its acceleration $\ddot{x}$ is given by $\ddot{x} = -4x + 3$. What are the centre and period of motion?

(A) centre of motion = 3, period = $\frac{\pi}{2}$

(B) centre of motion = -3, period = π

(C) centre of motion = $\frac{3}{4}$, period = π

(D) centre of motion = $\frac{3}{4}$, period = $\frac{\pi}{2}$

7. It is given that $z = 2 + i$ is a root of $z^3 + az^2 - bz + 5 = 0$, where a and b are real numbers.

What is the value of a ?

(A) –5

(B) –3

(C) 3

(D) 5

8. Which integral has the smallest value?

(A) $\displaystyle\int_0^{\frac{\pi}{4}} (\sin x)^2 \, dx$

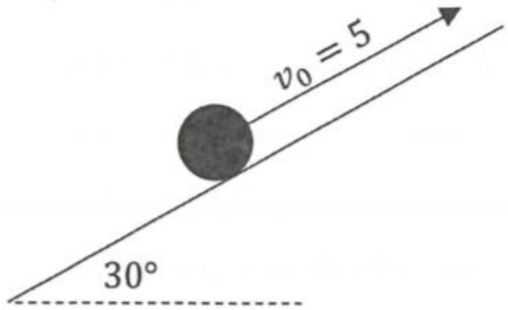
(B) $\displaystyle\int_0^{\frac{\pi}{4}} (\cos x)^2 \, dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \cos x dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \tan x dx$

9. A ball of mass m kg is observed rolling up a frictionless ramp, inclined at an angle of 30° to the horizontal. When first observed, the ball's velocity is 5 m/s.

Through what further distance will the ball roll up the ramp?



(A) $\sqrt{3}$ metres

(B) $m\sqrt{3}$ metres

(C) 2.5 metres

(D) $2.5m$ metres

10. What value of a will minimise the integral $\int_0^1 \left(x^2 - a\right)^2 dx$?

(A) $a = \frac{1}{2}$

(B) $a = \frac{1}{\sqrt{2}}$

(C) $a = \frac{4}{45}$

(D) $a = \frac{1}{3}$

End of Section I

Section II

90 marks
Attempt questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a separate writing booklet.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given $w = 2 + 5i$ and $z = 4 - 3i$, evaluate

(i) $\left| w + \overline{z} \right|$ **2**

(ii) $\left(w + \overline{z} \right)\left(\overline{w} + z \right)$ **2**

(b) Find the square roots of $15 - 8i$. **3**

(c) Use the substitution $t = \tan \dfrac{\theta}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \sin\theta + \cos\theta}$. **4**

(d) (i) Write down all the roots of $z^7 - 1 = 0$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. **2**

(ii) Hence, prove that $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7} = -\dfrac{1}{2}$ **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Use integration by parts to find $\int x3^x\,dx$. **3**

(b) By writing $\dfrac{8-2x}{(1+x)\left(4+x^2\right)}$ in the form $\dfrac{A}{1+x}+\dfrac{Bx+C}{4+x^2}$, evaluate $\displaystyle\int_0^4 \frac{8-2x}{(1+x)\left(4+x^2\right)}dx$. **4**

(c) On the same Argand diagram, draw a neat sketch of $|z-4-4i|=2$ and $\arg(z)=\dfrac{\pi}{4}$. **4**

Hence write down all the values of z which satisfy simultaneously

$|z-4-4i|=2$ and $\arg(z)=\dfrac{\pi}{4}$.

(d) Find the scalar projection of the vector $\underset{\sim}{u}=\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ onto $\underset{\sim}{v}=4\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}$. **2**

(e) Given $\underset{\sim}{a}=\begin{pmatrix}-1\\3\\4\end{pmatrix}$ and $\underset{\sim}{b}=\begin{pmatrix}-2\\1\\-4\end{pmatrix}$, and $\underset{\sim}{a}-\underset{\sim}{b}+2\underset{\sim}{c}=\underset{\sim}{0}$, find $\underset{\sim}{c}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Suppose a and b are positive integers, where a is even and b is odd. Show that the sum of a and b is odd. **1**

For positive integers m and n, consider the propositions:

p: $m+n$ is even

q: m and n are both even

r: m and n are both odd

(ii) Write in symbolic form the compound proposition P where **1**

P: If $m+n$ is even, then m and n are both even or m and n are both odd.

You may use $\vee$ for 'or' and $\wedge$ for 'and' where appropriate.

(iii) Write both in symbolic form and as an English sentence, the contrapositive to the proposition P in part (ii). **2**

(iv) Hence explain why the proposition P is true. **1**

(b) If x and y are positive integers with $x > y$, prove that $6(x+y)^2 - 2(x-y)^2$ is divisible by four. **1**

(c) Let $I_1 = \int_0^{\pi} \frac{x \sin x dx}{1+\cos^2 x}$ and $I_2 = \int_0^{\pi} \frac{(\pi - x)\sin x dx}{1+\cos^2 x}$.

(i) Using the substitution $u = \pi - x$ show that $I_1 = I_2$. **2**

(ii) Hence, or otherwise, evaluate I_1. **2**

Question 13 continues on page 10

Question 13 (continued)

(d) A particle is travelling in a straight line. Its displacement, x cm, from O at a given time t seconds after the start of its motion is given by $x = 3 + \sin^2 t$.

(i) Prove that the particle is undergoing simple harmonic motion. **2**

(ii) Find the period of the motion. **1**

(iii) Find the total distance travelled by the particle in the first π seconds. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove $\log_3 7$ is irrational. **2**

(b) The scalar product of $\underset{\sim}{i} - 2\lambda \underset{\sim}{j} - \underset{\sim}{k}$, and the sum of $\underset{\sim}{i} - \lambda \underset{\sim}{k}$ and $\lambda \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, is six. Find λ. **2**

(c) Prove by mathematical induction $(2n)! < (n!)^2 \times 4^{n-1}$ for all positive integers $n \geq 5$. **4**

(d) Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$.

(i) Prove that if $\overrightarrow{OD} = \frac{1}{5}\overrightarrow{OC}$ then D lies on AB. **2**

(ii) Is the point D closer to point A or point B? Justify your answer. **1**

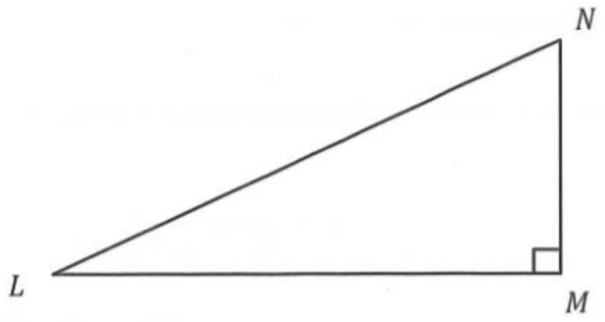
(e) (i) Given $z = \cos\theta + i\sin\theta$, prove that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$, where $n \in \mathbb{Z}$. **1**

(ii) Hence, by considering the expansion of $\left(z - \dfrac{1}{z}\right)^5$, find the values of a, b and c such that $\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



In ΔLMN, let $\overrightarrow{LM} = \underset{\sim}{a}$ and $\overrightarrow{MN} = \underset{\sim}{b}$. **3**

By finding an expression for the side length LN in terms of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, or otherwise, prove that $\left|\overrightarrow{LN}\right|^2 = \left|\overrightarrow{LM}\right|^2 + \left|\overrightarrow{MN}\right|^2$.

(b) By evaluating the integral, show that

$$\int x^2\sqrt{1-x^2}\,dx = \frac{\sin^{-1}(x)+\sqrt{1-x^2}\left(2x^3-x\right)}{8}+C$$ **3**

(c) It is given that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ for $k \in \mathbb{N}$.

(i) Prove that $\dfrac{1}{(k+1)^2} < \dfrac{1}{k(k+1)}$. **1**

(ii) If $x_1, x_2, x_3, \ldots, x_n$ are positive integers, not necessarily consecutive, such that **3**

$1 < x_1 < x_2 < x_3 < \ldots < x_n$, prove that $\dfrac{1}{x_1^2} + \dfrac{1}{x_2^2} + \dfrac{1}{x_3^2} + \ldots + \dfrac{1}{x_{n-1}^2} < 1$.

Question 15 continues on page 13

Question 15 (continued)

(d) A particle of mass m kg is moving vertically downwards in a medium which exerts a resistive force equal to mkv^2, where v is the particle's speed and $k > 0$ is a positive constant. The particle is released from rest at O and its terminal velocity is U.

(i) By applying Newton's 2nd Law, show that the distance it has fallen below O is given by: **2**

$$x = \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|.$$

(ii) Prove that the time taken, T, for the particle to fall from O to when its velocity is half of its terminal velocity, U, is given by: **3**

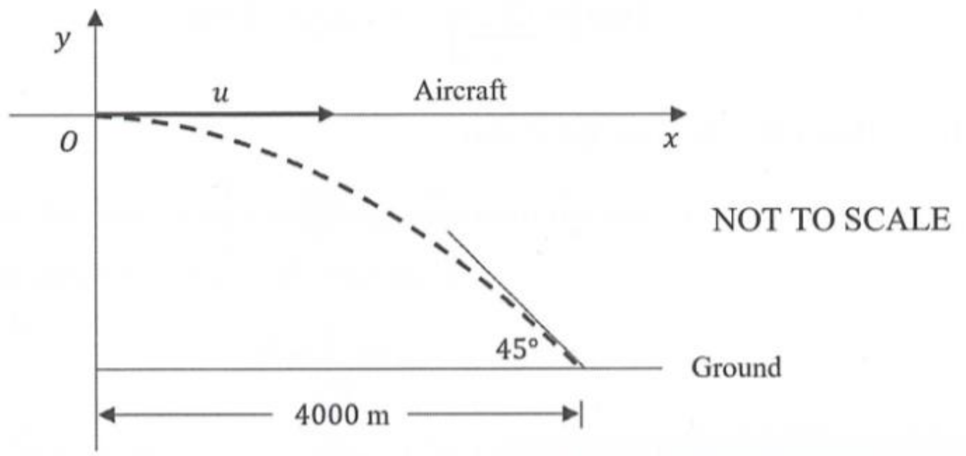
$$T = \frac{U}{2g}\ln 3.$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) An aircraft flying horizontally at u m/s delivers an emergency medical supply package that hits the ground 4000 m away, measured horizontally. The package experiences an air resistance of $0.1v$ where v is the velocity at time t and g is the acceleration due to gravity.

The package hits the ground at an angle of 45° to the horizontal.



The horizontal and vertical components of acceleration are given by the differential equations $\ddot{x} = -0.1\dot{x}$ and $\ddot{y} = -g - 0.1\dot{y}$ respectively.

(i) By solving these differential equations, show that t seconds after release, the position vector, $\underset{\sim}{r}(t)$, is given by: **5**

$$\underset{\sim}{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 10u(1-e^{-0.1t}) \\ 100g(1-e^{-0.1t}) - 10gt \end{pmatrix}$$

(ii) Using g = 10 m/s^2, find the time when the package hits the ground. Give your answer correct to 1 decimal place. **3**

Question 16 continues on page 15

Question 16 (continued)

(b) The lengths of the sides of a right triangle are a, b and c, where c is the length of the hypotenuse. Prove the Pythagorean Inequality Theorem: $a^3 + b^3 < c^3$. **2**

(c) (i) Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2}\, dx$ where n is a positive integer, **2**

prove that $I_{2n+1} = \frac{e}{2} - nI_{2n-1}$.

(ii) Hence, or otherwise, prove that $2\int_0^1 x^{2n-1}\left(1 + x^2\right) e^{x^2}\, dx \le e$ for $n \in \mathbb{N}$. **3**

End of Paper

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

SECTION I

Comments: 1. D 6. C, 2. A 7. B, 3. C 8. A, 4. C 9. C, 5. A 10. D

1. Let p, q & r be the statements

p: An animal is a bird

q: An animal can fly

r: An animal can swim

The compound statement in symbolic form is $p \to (q \vee r)$

Its contrapositive is:

$$\sim(q \vee r) \to \sim p$$

$$\Rightarrow \quad \sim q \wedge \sim r \to \sim p$$

Comments: De Morgan's Laws

$\Rightarrow$ If an animal cannot fly and (if an animal) cannot swim, then the animal is not a bird.

Of the alternatives (D) is correct.

Recall, in logic, 'or' as a conjunction has an inclusive sense, and this allows for the possibility that the animal can swim when it can't fly and vice versa.

2.
$$\frac{100\ {}^{x}C_{100}}{{}^{x-1}C_{99}} = \frac{100 \cdot \frac{x!}{(x-100)!\,100!}}{(x-1)!/(x-100)!\,99!}$$

$$= \frac{100\ x!}{(x-100)!\,100!} \times \frac{99!\,(x-100)!}{(x-1)!}$$

$$= \frac{x \times 100!\,(x-1)!\,(x-100)!}{100!\,(x-1)!\,(x-100)!} \text{ or } x.$$

So we have $x = \dfrac{1}{x^{99}}$

$$\Rightarrow \quad x^{100} - 1 = 0$$

Solutions given by $x = \operatorname{cis}\left(\dfrac{2k\pi}{100}\right)$

Comments: $k = 0, 1, \ldots, 99$

Real sol'ns correspond to $k = 0$ & $k = 50$ only, that is, $x = 1$ & $x = -1$.

However ${}^{n}C_{r}$ is only defined for $n \geq r$, hence, ${}^{x}C_{100}$ is only defined for $x \geq 100$; $x \in \mathbb{N}$. So the original equation has no real solutions. $\therefore$ (A)

3. If θ is the angle between $\underset{\sim}{u}$ & $\underset{\sim}{v}$, then $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}\right)$

$$= \cos^{-1}\left(\frac{1}{\sqrt{6} \cdot \sqrt{14}}\right) \text{ or } \cos^{-1}\left(\frac{1}{\sqrt{84}}\right)$$

$$= 83.736\ldots^\circ \approx 84^\circ \quad \therefore \text{(C)}$$

4. $\overrightarrow{AB} = (2, -14, 2)$ & $\overrightarrow{CD} = (2, -14, 0)$

$$\operatorname{Proj}_{\overrightarrow{CD}} \overrightarrow{AB} = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\overrightarrow{CD} \cdot \overrightarrow{CD}} \cdot \overrightarrow{CD} \qquad \therefore \text{(C)}$$

$$= \frac{198}{198}(2, -14, 0) \text{ or } 2\underset{\sim}{i} - 14\underset{\sim}{j}$$

5. $2 > -1 \longrightarrow \frac{1}{2} < -1$ is false with $2, -1 \in \mathbb{Z} \setminus \{0\}$.

Here p: $2 > 1$ & q: $\frac{1}{2} < -1$

The truth value of p is true or 1 & the truth value of q is false or 0. Consequently, the truth value of the implication $p \to q$ is false.

So (A)

B: F F $\Rightarrow$ implication is true
inputs

C: T T $\Rightarrow$ implication is true

D: F T $\Rightarrow$ implication is true

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

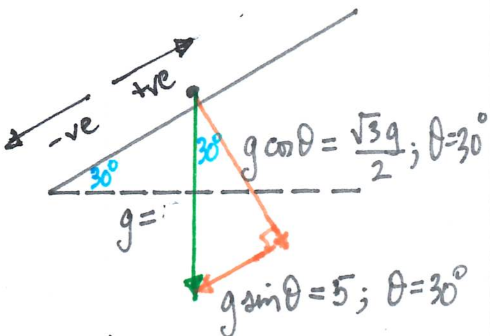
Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
6. $\ddot{x} = -4\left(x-\frac{3}{4}\right)$ or $= -(2)^2\left(x-\frac{3}{4}\right)$ $\therefore x_0 = 3/4$ $\therefore$ (C) $\tau = 2\pi/2$ or π	Of type $\ddot{x}=-n^2(x-x_0)$ $\tau = \frac{2\pi}{n}$ x_0 = centre of motion	8. For $x \in \left[0, \frac{\pi}{4}\right]$ all integrands are non-negative with each equally zero only when $x=0$. So all integrals, when evaluated are positive. Now $\sin x \le \cos x$ on $[0, \pi/4]$ with equality only achieved at $x=\pi/4$. $\therefore \sin^2 x \le \cos^2 x \cdots$ (I) & $\sin x \cdot \sin x \le \cos x \cdot \sin x$ $\Rightarrow \sin^2 x \le \sin x \cdot \cos x \cdots$ (II)	as $\sin x \ge 0$ on $\left[0, \frac{\pi}{4}\right]$
7. Product of roots $=-5$ Given $2+i$ is a root, then $2-i$ is also a root as the polynomial equation satisfies the criteria of the CONJUGATE ROOT THEOREM. If the remaining root is α, then $(2+i)(2-i)\cdot\alpha = -5$ $\Rightarrow 5\alpha = -5$ or $\alpha = -1$. So roots are $2 \pm i, -1$ & sum of roots, $\sum \alpha_i = -a$ $\Rightarrow (2+i)+(2-i)+(-1) = -a$ $\Rightarrow a = -3$ $\therefore$ (B)	Relationships between roots & coefficients Relationships between roots & relationships	Finally $\frac{1}{\sqrt{2}} \le \cos x \le 1$ on $\left[0, \frac{\pi}{4}\right]$ So, $1 \le \frac{1}{\cos x} \le \sqrt{2}$ $\therefore \sin x \cdot \tan x = \frac{\sin^2 x}{\cos x}$ $= \sin^2 x \cdot \frac{1}{\cos x}$	
ALTERNATIVE: Coefficients of polynomial are real and so $\overline{2+i}$ is also a root Noting $(z-\alpha)(z-\bar{\alpha}) = z^2 - (\alpha+\bar{\alpha})z + \alpha\bar{\alpha}$ $= z^2 - 2\mathrm{Re}(\alpha)\cdot z + \|\alpha\|^2$ For $\alpha = 2+i$, then z^2-4z+5 is a factor of z^3+az^2-bz+5. So $z^3+az^2-bz+5 \equiv (z-c)(z^2-4z+5)$ where c is the remaining root $\therefore -5c = 5$ or $c=-1$. Hence, sum of roots $=(2+i)+(2-i)+(-1)$ $= -a \Rightarrow a = -3$. So (B)	Conjugate Root theorem Factor theorem $\sum \alpha_i = -\frac{a}{1}$	Upon multiplying the inequality $1 \le \frac{1}{\cos x} \le \sqrt{2}$ by $\sin^2 x \ge 0$ $\Rightarrow \sin^2 x \le \sin x \cdot \tan x \le \sqrt{2}\cos^2 x$. $\cdots$ (III) From inequalities (I), (II) & (III), we conclude that $\int_0^{\pi/4} \sin^2 x \cdot dx$ has the least value. $\therefore$ (A) Graphical Approach available.	$x \in \left[0, \frac{\pi}{4}\right]$

VIRILE AGITUR

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

9.



Taking the positive sense in the direction of motion; here, up the ramp, then:

$\ddot{x} = -g\sin\theta$

$\ddot{x} = -5$

Comments: $\theta = 30°$, $g = 10\,m/s^2$

$\Rightarrow \dot{x} = c_1 - 5t$ & $\dot{x} = 5\,m/s$ when $t = 0$

$\Rightarrow \dot{x} = 5 - 5t \qquad \therefore c_1 = 5$

$\Rightarrow x = 5t - \frac{5}{2}t^2 + c_2$ & $x = 0$ when $t = 0$ $\Rightarrow c_2 = 0$

So $\begin{cases} \dot{x} = 5 - 5t \\ \& \\ x = 5t - \frac{5}{2}t^2. \end{cases}$

Object comes to rest when $t = 1$

Comments: $\dot{x} = 0$

& $x(1) = 5 - 5/2$ or 2.5

Hence, ball rolls a further 2.5m up ramp. $\therefore$ (C)

10.

$$\int_0^1 (x^2 - a)^2\,dx$$

$$= \int_0^1 (x^4 - 2ax^2 + a^2)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{2a}{3}x^3 + a^2x\right]_0^1$$

$$= a^2 - \frac{2a}{3} + \frac{1}{5}$$

$$= \left(a - \frac{1}{3}\right)^2 + \frac{1}{5} - \frac{1}{9} \quad \text{OR}$$

$$\left(a - \frac{1}{3}\right)^2 + \frac{4}{45}$$

This algebraic expression is minimised when $a = 1/3$.

ALTERNATIVE:

Let $b = a^2 - \frac{2a}{3} + \frac{1}{5}$.

b is minimised when $\frac{db}{da} = 0$.

This implies $2a - \frac{2}{3} = 0$

$\Rightarrow a = 1/3$.

$\therefore$ (D)

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q11 SECTION II

(a) (i) $\omega = 2+5i$; $z = 4-3i$

$\therefore |\omega+\bar{z}| = |(2+5i)+(4+3i)|$

$= |6+8i|$ — ✓ process

$= \sqrt{6^2+8^2}$ or 10 — ✓ answer

(ii) Noting $\overline{\omega+\bar{z}} = \bar{\omega}+\bar{\bar{z}}$

$= \bar{\omega}+z$ — Properties of Conjugate

Then $(\omega+\bar{z})(\bar{\omega}+z) = (\omega+\bar{z})(\overline{\omega+\bar{z}})$

$= |\omega+\bar{z}|^2$ — $z\bar{z} = |z|^2$

$= 10^2$ or 100 — ✓ process ✓ answer

ALTERNATIVE:

$(\omega+\bar{z})(\bar{\omega}+z) = (6+8i)(6-8i)$

$= 6^2+8^2$ or 100 — $-8i^2 = 8$

(b) Let $x+iy$ be a square root of $15-8i$. Then $(x+iy)^2 = 15-8i$

$\Rightarrow \left.\begin{aligned} x^2-y^2 &= 15 \\ xy &= -4 \end{aligned}\right\}$ upon equating real & imaginary parts — ✓ simultan. equations

By inspection $x = \pm 4$ & $y = \mp 1$

Hence square roots are $4-i$ & $-4+i$, i.e., $\pm(4-i)$ — ✓ correct ✓ answers

ALTERNATIVE: Substitute $y = -\frac{4}{x}$ into $x^2-y^2 = 15$

$\therefore x^2 - \frac{16}{x^2} = 15$

$\Rightarrow x^4-15x^2-16 = 0$ — $\otimes$ by x^2

$\Rightarrow (x^2-16)(x^2+1) = 0$

$\Rightarrow x = \pm 4$ only ; $x \in \mathbb{R}$

whereupon $y = \mp 1$ respectively

(c) $t = \tan\theta/2$, then $d\theta = \frac{2\,dt}{1+t^2}$

ALTERNATIVE: $\frac{dt}{d\theta} = \frac{1}{2}\sec^2\theta/2$

$= \frac{1}{2}(1+\tan^2\theta/2)$ — Pythagorean Identity

$\frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$

$\Rightarrow \frac{d\theta}{dt} = \frac{2}{1+t^2}$ or equivalently,

$d\theta = \frac{2\,dt}{1+t^2}$. — ✓ correct differential

When $\begin{cases} \theta = 0, & t = 0 \\ \theta = \frac{\pi}{2}, & t = 1 \end{cases}$

$\sin\theta = \frac{2t}{1+t^2}$ & $\cos\theta = \frac{1-t^2}{1+t^2}$ — } 1 mark

$\therefore$ integral becomes:

$$\int_0^1 \frac{2/(1+t^2)\;dt}{1+2t/(1+t^2)+(1-t^2)/(1+t^2)}$$

$$= \int_0^1 \frac{2\,dt}{2+2t} \text{ OR } \int_0^1 \frac{dt}{1+t}$$ ← 1 mark for correct integrand

$$= \ln|1+t|\Big|_0^1 \text{ or } \ln 2$$ — 1 mark for correct ans.

(d) (i) Roots are given by:

$z_k = r^{1/n} e^{i\left(\frac{\theta+2k\pi}{n}\right)}$, $k = 0,1,2,\dots,6$ — $n = 7$, $r = 1$, $\theta = \operatorname{Arg}(1) = 0$

$= e^{i\left(\frac{2k\pi}{7}\right)}$; — ✓ progress

$z_0 = 1$

$z_1 = e^{i\left(\frac{2\pi}{7}\right)}$

$z_2 = e^{i\left(\frac{4\pi}{7}\right)}$

$z_3 = e^{i\left(\frac{6\pi}{7}\right)}$

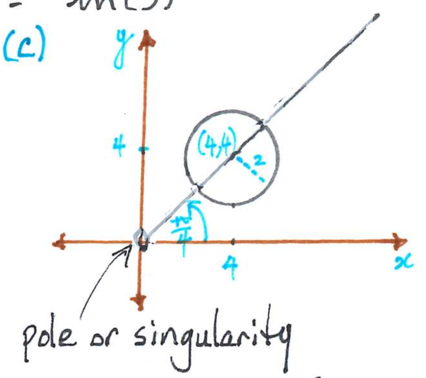
$z_4 = e^{i\left(\frac{8\pi}{7}\right)} = e^{i\left(-\frac{6\pi}{7}\right)}$

$z_5 = e^{i\left(\frac{10\pi}{7}\right)} = e^{i\left(-\frac{4\pi}{7}\right)}$

$z_6 = e^{i\left(\frac{12\pi}{7}\right)} = e^{i\left(-\frac{2\pi}{7}\right)}$

✓✓ for correct answers with principal arguments

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q11 (d) (ii) $\sum \alpha_i = 0$ $\therefore 1 + e^{i(\frac{2\pi}{7})} + e^{i(-\frac{2\pi}{7})} + e^{i(\frac{4\pi}{7})} + e^{i(-\frac{4\pi}{7})} + e^{i(\frac{6\pi}{7})} + e^{i(-\frac{6\pi}{7})} = 0$ $\Rightarrow 1 + 2\cos(\frac{2\pi}{7}) + 2\cos(\frac{4\pi}{7}) + 2\cos(\frac{6\pi}{7}) = 0$ $\Rightarrow \cos(\frac{2\pi}{7}) + \cos(\frac{4\pi}{7}) + \cos(\frac{6\pi}{7}) = -\frac{1}{2}$ Question 12 (a) Let $u = x \Rightarrow du = dx$ & let $dv = 3^x dx \Rightarrow v = \frac{3^x}{\ln(3)}$. $\int u\,dv = uv - \int v\,du$ $\therefore \int x3^x dx = \frac{x3^x}{\ln(3)} - \frac{1}{\ln(3)}\int 3^x dx$ $= \frac{x3^x}{\ln(3)} - \frac{3^x}{(\ln(3))^2} + C$ OR $\frac{3^x}{\ln(3)}\left[x - \frac{1}{\ln(3)}\right] + C$ N.B. $\int a^x dx = \frac{a^x}{\ln(a)}$ for $a > 0$ & $a \neq 1$. (b) $\frac{8-2x}{(1+x)(4+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{4+x^2}$ $\Rightarrow 8-2x \equiv A(4+x^2) + (Bx+C)(1+x)$ When $x = -1$: $10 = 5A \Rightarrow A = 2$. Equating coefficients of x^2: $0 = A + B \Rightarrow B = -2$ Equating constants: $8 = 4A + C \Rightarrow C = 0$	Relationships between roots and coefficients $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ ✓✓ process ALTERNATIVE STRATEGIES AVAILABLE - SEE SUPPLEMENT Integration by Parts ✓ attempting to use IP ✓✓ substantial progress ✓✓✓ correct solution ✓ for at most two correct values ✓✓ for all three correct values	$\therefore \int_0^4 \frac{8-2x}{(1+x)(4+x^2)}$ $= \int_0^4 \left(\frac{2}{1+x} - \frac{2x}{4+x^2}\right) dx$ $= 2\ln\lvert 1+x\rvert - \ln(4+x^2)\Big\rvert_0^4$ $= 2\ln(5) - \ln(20) + \ln(4)$ $= 2\ln(5) - \ln(4) - \ln(5) + \ln(4)$ $= \ln(5)$ (c)  Cartesian form of circle is $(x-4)^2 + (y-4)^2 = 4$ ... (I) Cartesian form of ray is $y = x;\ x > 0$ ... (II) Solving simultaneously, $2(x-4)^2 = 4$ $\Rightarrow x = 4 \pm \sqrt{2}$ $\Rightarrow y = 4 \pm \sqrt{2}$ Hence $z = (4 \pm \sqrt{2}) + i(4 \pm \sqrt{2})$ or $z = (4 \pm \sqrt{2}, 4 \pm \sqrt{2})$	✓ correct integration $\ln(ab) = \ln(a) + \ln(b)$ ✓ correct answer ✓ correct sketch of circle ✓ correct sketch of ray EXCLUDING origin ✓ for simultaneous system ✓ for correct answers

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q12(d) Scalar Projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ is given by $\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{v}\rvert}$ where $\underset{\sim}{u}\cdot\underset{\sim}{v} = 4+8+7$ or 19 & $\lvert\underset{\sim}{v}\rvert = \sqrt{4^2+(-4)^2+7^2}$ or 9 $\therefore$ scalar projection $= \frac{19}{9}$	✓ for correct dot product ✓ correct answer	(iii) We have $p \to (q \vee r)$ & its contrapositive is :- $\sim(q \vee r) \to \sim p$ $\Leftrightarrow \sim q \wedge \sim r \to \sim p$	$\vee \equiv$ or $\wedge \equiv$ and ✓ correct contrapositive - symbolic de Morgan's Laws
(e) $\begin{pmatrix}-1\\3\\4\end{pmatrix} - \begin{pmatrix}-2\\1\\-4\end{pmatrix} + 2\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\Rightarrow \begin{cases}2x+1=0\\2y+2=0\\2z+8=0\end{cases} \Rightarrow x=-\frac{1}{2}$, $y=-1$ & $z=-4$. $\therefore \underset{\sim}{c} = \begin{pmatrix}-\frac{1}{2}\\-1\\-4\end{pmatrix}$	$\underset{\sim}{c} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ ✓✓ for correct solution ✓ one careless error only appearing in solution	If m and n are NOT both even AND if m and n are NOT both odd, then m+n is NOT even. OR If one of m and n is even and the other is odd, then m+n is ODD.	✓ for correct contrapositive English form
Question 13 (a) (i) Let $a = 2k_1$; $k_1 \in \mathbb{N}$; let $b = 2k_2+1$; $k_2 \in \mathbb{N}$. Then $a+b = 2k_1 + (2k_2+1)$ $= 2(k_1+k_2)+1$ $= 2K+1$; $K = k_1+k_2$ which is ODD by definition	✓✓ for process Positive integers are closed under addition	(iv) This result was proven in part (i) and as an implication & its contrapositive are logically equivalent, proposition P is true.	✓ logical equivalence mandatory
(ii) p: m+n is even q: m and n are both even r: m and n are both odd $p \to (q \vee r)$	✓ correct answer	(b) $6(x+y)^2 - 2(x-y)^2$ $= 6x^2+12xy+6y^2-2x^2+4xy-2y^2$ $= 4x^2+16xy+4y^2$ $= 4(x^2+4xy+y^2)$ $= 4M$; $M = x^2+4xy+y^2 \in \mathbb{N}$ $\therefore 4 \mid \{6(x+y)^2-2(x-y)^2\}$ See Over for (c)	✓ for proof Positive integers closed under ⊗ and ⊕

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments
Q13(c)(i) For $I_1 = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$,	
let $u = \pi - x$ $(x = \pi - u)$. When $\begin{cases} x=0 \\ u=\pi \end{cases}$ &	✓✓ for process
$\Rightarrow du = -dx$ or $dx = -du$. When $\begin{cases} x=\pi \\ u=0 \end{cases}$	$\sin(\pi-\theta) = \sin\theta$;
$\therefore I_1 = -\int_{\pi}^{0} \frac{(\pi-u)\cdot\sin(\pi-u)}{1+\cos^2(\pi-u)}\,du$	$\cos(\pi-\theta) = -\cos\theta$
$= \int_0^{\pi} \frac{(\pi-u)\cdot\sin u\,du}{1+\cos^2 u}$	$-\int_a^b f = \int_b^a$
$= \int_0^{\pi} \frac{(\pi-x)\cdot\sin x\,dx}{1+\cos^2 x}$	changing dummy variable
$= I_2$.	
(ii) $I_1 + I_2 = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$	Formally, integration by substitution with $u=\cos x$
$= -\pi\int_0^{\pi} \frac{-\sin x\,dx}{1+\cos^2 x}$	$-\pi\int_1^{-1} \frac{du}{1+u^2}$
$= -\pi\left[\tan^{-1}(\cos x)\right]_0^{\pi}$	or $\pi\int_{-1}^{1} \frac{du}{1+u^2}$
$= -\pi\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$	$= 2\pi\int_0^{1} \frac{du}{1+u^2}$ integrand EVEN
$= -\pi\left[-2\tan^{-1}(1)\right]$	$= \frac{\pi^2}{2}$
$= 2\pi \cdot \frac{\pi}{4}$ or $\frac{\pi^2}{2}$ ✓✓ for process	

Suggested Solution (s)	Comments
(d)(i) Required to show that $\ddot{x} = -n^2(x - x_0)$	n = angular frequency; x_0 = centre of motion
Given $x = 3 + \sin^2 t$	
$\dot{x} = 2\sin t\cdot\cos t$ (chain rule)	
$\therefore \ddot{x} = 2\sin t\frac{d}{dt}(\cos t) +$	Product rule
$\Rightarrow 2\cos t\cdot\frac{d}{dt}(\sin t)$	
$= -2\sin^2 t + 2\cos^2 t$	
$= 2(1-\sin^2 t) - 2\sin^2 t$	
$= 2 - 4\sin^2 t$	
& $\sin^2 t = x - 3$	✓✓ for process
$\therefore \ddot{x} = 2 - 4(x-3)$	
$= 14 - 4x$	
$= -4\left(x - \frac{14}{4}\right)$	
OR $\ddot{x} = -(2)^2\left(x - \frac{7}{2}\right)$	
ALTERNATIVE:	
$x = 3 + \sin^2 t$	
$= 3 + \frac{1}{2}(1 - \cos 2t)$	
$= \frac{7}{2} - \frac{1}{2}\cos 2t$	
$\Rightarrow \dot{x} = \sin 2t$	
$\Rightarrow \ddot{x} = 2\cos 2t$	
& $\cos 2t = 7 - 2x$ from $x = \frac{7}{2} - \frac{1}{2}\cos 2t$	
$\therefore \ddot{x} = 2(7-2x)$	
$= -(2)^2\left(x - \frac{7}{2}\right)$	

VIRILE AGITUR

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q13 (d) (ii) $\tau = \frac{2\pi}{n}$ & $n = 2$ from (i) $\therefore \tau = \pi$ seconds (iii) Noting $0 \le \sin^2 t \le 1$, then $3 \le x \le 4$ with centre of motion obviously $x = 3.5$. Initially, $x = 3$ (left extremity of interval & since $\tau = \pi$, particle completes 1 cycle, i.e., travels twice the path length. Hence, total distance travelled is 2 cm. Question 14: (a) Suppose, to the contrary, $\log_3 7 = \frac{p}{q}$; $(p, q) = 1$. $\Rightarrow 3^{p/q} = 7$ or equivalently $3^p = 7^q$ As $\log_3 7 > \log_3 3 = 1$, then $p, q \in \mathbb{N}$. The Fundamental Theory of Arithmetic states that the prime-factor decomposition is UNIQUE for any positive integer greater than 1. If $N = 3^p$, i.e., if $N = \underbrace{3 \times 3 \times \dots \times 3}_{p \text{ times}}$; $N \in \mathbb{N}$ as +ve integers closed under mult^{n}.	✓ correct answer ✓✓ for process ALGEBRAIC METHODS AVAILABLE p & q co-prime def^{n} of a logarithm log is an increasing.	then, since $3^p = 7^q$, we have two prime factor decompositions for the positive integer N, which is impossible. Hence, we have a contradiction. ALTERNATIVE: FROM $3^p = 7^q$ $\Rightarrow 3(3^{p-1}) = 7^q$ $\Rightarrow 3 \mid 7^q$ $\Rightarrow 3 \mid 7$ Consequence of Euclid's Lemma However $3 \nmid 7$ & so we have a contradiction. Hence, our supposition is false; namely, $\log_3 7$ is <u>not</u> rational (b) $(\underset{\sim}{i} - 2\lambda \underset{\sim}{j} - \underset{\sim}{k}) \cdot ((\underset{\sim}{i} - \lambda \underset{\sim}{k}) + (\lambda \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}))$ $= (\underset{\sim}{i} - 2\lambda \underset{\sim}{j} - \underset{\sim}{k}) \cdot ((1+\lambda)\underset{\sim}{i} + 2\underset{\sim}{j} + (-\lambda - 1)\underset{\sim}{k})$ Since the result is 6, we have: $(1+\lambda) - 4\lambda + (\lambda + 1) = 6$ $\Rightarrow 2 - 2\lambda = 6$ or $\lambda = -2$ (c) Basis Step: When $n = 5$, $(2n!) = 10!$ $= 3\,628\,800$ & $(n!)^2 4^{n-1} = (5!)^2 \times 4^4$ $= (120)^2 \times 256$ $= 3\,686\,400$ Clearly $3\,628\,800 < 3\,686\,400$	✓ for attempting proof by contradiction & eliminating logarithm for ✓✓ coherent proof. ✓ for solution with one error only ✓✓ for correct solution ✓ for basis step – calculations MUST be shown

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q14 (c) Cont. INDUCTIVE STEP Assume $(2n)! < (n!)^2 \times 4^{n-1}$ for some $n = k$; $k \in \mathbb{N} \setminus \{1,2,3,4\}$. Hence $P(k): (2k)! < (k!)^2 \cdot 4^{k-1}$ When $n = k+1$, $(2(k+1))! = (2k+2)!$ $= (2k+2)(2k+1)(2k)\cdot(2k-1)\cdot \cdots 3\cdot 2\cdot 1$ $= (2k+2)(2k+1)\cdot[(2k)!]$ $< (2k+2)(2k+1)\cdot(k!)^2\cdot 4^{k-1}$ $< (2k+2)(2k+2)\cdot(k!)^2\cdot 4^{k-1}$ $= 4[(k+1)!]^2\cdot(k!)^2\cdot 4^{k-1}$ $= ((k+1)!)^2\cdot 4^k$ Hence, $P(k) \Longrightarrow P(k+1)$ As $P(1)$ is true, conclude that $(2n)! < (n!)^2 \times 4^{n-1}$ $\forall n \in \mathbb{N} \setminus \{1,2,3,4\}$ by induction (d) (i) Equation of line AB given by $r = a + \lambda(b - a)$ i.e. $r = (1-\lambda)a + \lambda b$; $\lambda \in \mathbb{R}$ If D lies on AB, then we should be able to find a value for λ such that $r = \frac{1}{5}(3a + 2b)$	INDUCTIVE STEP ✓✓✓ for correct solution using inductive hypothesis $2k+1 < 2k+2$ i.e. $1 < 2$ ✓✓ for minor logic errors and/or omissions ✓ states IH & attempts to use it to simplify P(k+1) ✓ for equation of line AB + ✓ $\lambda = \frac{2}{5}$ $\vec{OD} = \frac{1}{5}\vec{OC}$	That is: $\frac{3}{5}a + \frac{2}{5}b = (1-\lambda)a + \lambda b$ Clearly, $\lambda = \frac{2}{5}$ works. (ii) $\vec{OD} = a + \frac{2}{5}(b - a)$ $\|b - a\|$ measures the length of AB. $\lambda = 0$ corresponds to the point A. $\lambda = 1$ corresponds to the point B. $\therefore 0 < \lambda < 1$ corresponds to points in the interval [AB] measured from A to B. As $\frac{2}{5} < \frac{1}{2}$, D is closer to A, noting $\lambda = \frac{1}{2}$ would correspond to the midpoint M of [AB]. ALTERNATIVE: $b - a$ can be construed as the vector from A to B. Consequently, $a + \frac{1}{2}(b - a)$ would give the midpoint of AB. As $\frac{2}{5} < \frac{1}{2}$, then the point D, with position vector $\vec{OD} = a + \frac{2}{5}(b - a)$ would be closer to A than B.	Corresponding scalar eq'ns $\begin{cases} 1-\lambda = \frac{3}{5} \\ \lambda = \frac{2}{5} \end{cases}$ are consistent ✓ for coherent argument

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments
Q14(e) (i) By de Moivre's theorem: $z^n = (\text{cis}\,\theta)^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ & $z^{-n} = (\text{cis}\,\theta)^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta = \cos n\theta - i\sin n\theta$	sin ODD & cos EVEN
$\therefore\ z^n - z^{-n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$ i.e., $z^n - z^{-n} = 2i\sin n\theta$, as required.	✓ correct solution
(ii) $\left(z-\frac{1}{z}\right)^5 = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 - 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 - \left(\frac{1}{z}\right)^5$ $= z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$ $= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$	✓ Expands $(z-\frac{1}{z})^5$ correctly + ✓ rearranges binomial expansion into conjugate pairs and
$\therefore\ \left(z-\frac{1}{z}\right)^5 = 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta$ ...(I) On the other hand, $\left(z-\frac{1}{z}\right)^5 = (2i\sin\theta)^5$ OR $32i^5\sin^5\theta$ ...(II) Equating (I) & (II) $32i^5\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ $\Rightarrow\ 32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$ or equivalently, $\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$ Hence, $a = 1/16$, $b = -5/16$ & $c = 5/8$.	$i^5 = i$ use result in (i) successfully + ✓ for correct values for a, b & c.

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments

Question 15

(a) $|\overrightarrow{LN}| = |\overrightarrow{LM} + \overrightarrow{MN}|$

$= |\underset{\sim}{a} + \underset{\sim}{b}|$ ✓ for correct expression

$|\overrightarrow{LN}|^2 = |\underset{\sim}{a} + \underset{\sim}{b}|^2$

$= (\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= |\underset{\sim}{a}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2$ using properties of dot product

By construction $\overrightarrow{LM}$ & $\overrightarrow{MN}$ are orthogonal & so $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\therefore |\overrightarrow{LN}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$, i.e., $|\overrightarrow{LN}|^2 = |\overrightarrow{LM}|^2 + |\overrightarrow{MN}|^2$, as required

Comments: $\overrightarrow{LM} = \underset{\sim}{a}$, $\overrightarrow{MN} = \underset{\sim}{b}$; $|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a}$, $|\underset{\sim}{b}|^2 = \underset{\sim}{b} \cdot \underset{\sim}{b}$; + ✓✓ for coherent argument

(b) For $\int x^2\sqrt{1-x^2}\,dx$, let $x = \sin\theta$, i.e., $\theta = \sin^{-1}(x)$.

Then $dx = \cos\theta\, d\theta$ & the integral becomes :-

$$\int \sin^2\theta \cdot \sqrt{1-\sin^2\theta} \cdot \cos\theta \cdot d\theta = \int \sin^2\theta \cdot \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\int (2\sin\theta \cdot \cos\theta)^2 d\theta \text{ or } \frac{1}{4}\int \sin^2 2\theta \, d\theta \quad ; \sin^2 n\theta = \frac{1}{2}(1-\cos 2n\theta)$$

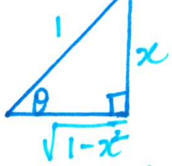
$$= \frac{1}{8}\int (1-\cos 4\theta)\,d\theta$$

$$= \frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right) + C$$

$$= \frac{1}{8}\left(\theta - \frac{2\sin 2\theta \cdot \cos 2\theta}{4}\right) + C$$

$$= \frac{1}{8}\left(\theta - \frac{4\sin\theta \cdot \cos\theta \cdot (1-2\sin^2\theta)}{4}\right) + C$$

$$= \frac{1}{8}\left(\sin^{-1}(x) - x \cdot \sqrt{1-x^2} \cdot (1-2x^2)\right) + C$$



$\begin{cases} \sin\theta = x \\ \cos\theta = \sqrt{1-x^2} \end{cases}$

or

$$\frac{\sin^{-1}(x) + \sqrt{1-x^2} \cdot (2x^3 - x)}{8} + C, \text{ as required.}$$

Comments: ✓✓✓ for correct sol'n; ✓✓ substantial progress; ✓ basic progress

Question 15(c)(ii)

VERSION 1: From $1 < x_1 < x_2 < \cdots < x_n$ with $x_i \in \mathbb{N}$ with x_i not necessarily equal to $x_{i-1}+1$, we note:

$$\left.\begin{array}{l} x_1 \geq 2 \\ x_2 \geq 3 \\ \vdots \\ x_n \geq n+1 \end{array}\right\} \therefore \begin{array}{l} \frac{1}{x_1} \leq \frac{1}{2} \\ \frac{1}{x_2} \leq \frac{1}{3} \\ \vdots \\ \frac{1}{x_n} \leq \frac{1}{n+1} \end{array}$$

✓✓✓ for argument
✓✓ for substantial progress
✓ for basic progress

In particular: $\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_{n-1}^2} \leq \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{n}\right)^2$.

So $\sum_{i=1}^{n-1} \left(\frac{1}{x_i}\right)^2 \leq \sum_{i=2}^{n} \left(\frac{1}{i}\right)^2$; Let $k+1 = i \Rightarrow \sum_{i=2}^{n} \left(\frac{1}{i}\right)^2 = \sum_{k=1}^{n-1} \frac{1}{(k+1)^2}$

$$\therefore \sum_{i=1}^{n-1} \left(\frac{1}{x_i}\right)^2 \leq \sum_{k=1}^{n-1} \frac{1}{(k+1)^2}$$

$$< \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \text{ using (i)}$$

$$= \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \begin{array}{l} 1 - \frac{1}{2} \; + \\ \frac{1}{2} - \frac{1}{3} \; + \\ \vdots \\ \frac{1}{n-2} - \frac{1}{n-1} \; + \\ \frac{1}{n-1} - \frac{1}{n} \\ \hline 1 - \frac{1}{n} \end{array}$$

Consequently,

$$\sum_{i=1}^{n-1} \left(\frac{1}{x_i}\right)^2 < 1 - \frac{1}{n} \; ; \; n \in \mathbb{N}$$

< 1, as required.

Question 15(c)(i) VERSION 1

$$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$$

✓ for correct argument

$\Leftrightarrow k(k+1) < (k+1)^2$; $k \in \mathbb{N}$

$\Leftrightarrow k^2 + k < k^2 + 2k + 1$

$\Leftrightarrow 0 < k+1$, which is true

VERSION 2 $\quad k+1 > k \; (>0)$

$\Rightarrow \frac{1}{k+1} < \frac{1}{k}$

$\Leftrightarrow \frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$ as $\frac{1}{k+1} > 0$

Question 15 (c) (ii)

VERSION 2: We have $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$ or equivalently $\frac{1}{k^2} < \frac{1}{(k-1)k}$

$= \frac{1}{k} - \frac{1}{k+1}$ $\qquad = \frac{1}{k-1} - \frac{1}{k}$.

So $\frac{1}{x_1^2} < \frac{1}{x_1 - 1} - \frac{1}{x_1}$ Now $x_1 > 1 \Rightarrow x_1 - 1 \geq 1$

$\leq 1 - \frac{1}{x_1}$ $\qquad \Rightarrow \frac{1}{x_1 - 1} \leq 1$ *

& $\frac{1}{x_2^2} < \frac{1}{x_2 - 1} - \frac{1}{x_2}$ with $x_2 > x_1 \Rightarrow x_2 - 1 \geq x_1$

$\Rightarrow \frac{1}{x_2 - 1} \leq \frac{1}{x_1}$

$\leq \frac{1}{x_1} - \frac{1}{x_2}$

$\vdots$

$\frac{1}{x_i^2} < \frac{1}{x_i - 1} - \frac{1}{x_i}$ with $x_i > x_{i-1} \Rightarrow x_i - 1 \geq x_{i-1}$

$\Rightarrow \frac{1}{x_i - 1} \leq \frac{1}{x_{i-1}}$

$\leq \frac{1}{x_{i-1}} - \frac{1}{x_i}$

$\therefore \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \dots + \frac{1}{x_{n-1}^2} < \left(1 - \frac{1}{x_1}\right) + \left(\frac{1}{x_1} - \frac{1}{x_2}\right) + \left(\frac{1}{x_2} - \frac{1}{x_3}\right) + \dots + \left(\frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}\right)$

$= 1 - \frac{1}{x_{n-1}}$

< 1 as $x_{n-1} > 1$; i.e., $x_{n-1} \in \mathbb{N} \setminus \{1\}$

✓✓✓ for argument
✓✓ for substantial progress
✓ for basic progress

* We have $\frac{1}{x_1^2} < \frac{1}{x_1 - 1} - \frac{1}{x_1} \implies \frac{1}{x_1^2} < 1 - \frac{1}{x_1}$

$\leq 1 - \frac{1}{x_1}$

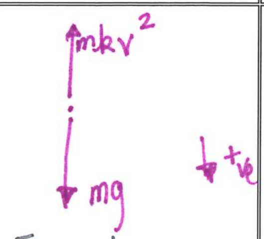
& in general: $\frac{1}{x_i^2} < \frac{1}{x_i - 1} - \frac{1}{x_i} \implies \frac{1}{x_i^2} < \frac{1}{x_{i-1}} - \frac{1}{x_i}$

$\leq \frac{1}{x_{i-1}} - \frac{1}{x_i}$

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments

Question 15 (d) (i) Taking the positive sense of all vector quantities in the direction of motion, i.e., vertically downwards, we can apply Newton's 2ND Law as follows:



Force diagram

$$m\ddot{x} = mg - mkv^2 \qquad \begin{cases} m = \text{mass (kg implied)} \\ g \text{ (gravitational constant)} \end{cases}$$

$$\Rightarrow \quad \ddot{x} = g - kv^2$$

$$\Rightarrow \quad v \cdot \frac{dv}{dx} = g - kv^2$$

$$\Rightarrow \int \frac{v\,dv}{g-kv^2} = \int dx$$

Variables Separable Approach

$$\Rightarrow -\frac{1}{2k}\int \frac{-2kv}{g-kv^2} = \int dx$$

$$\Rightarrow -\frac{1}{2k}\ln|g-kv^2| = x + C$$

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

& $x = 0$ when $v = 0 \Rightarrow C = -\frac{1}{2k}\ln(g)$

Consequently, $x = \frac{1}{2k}\ln(g) - \frac{1}{2k}\ln|g-kv^2|$

OR $x = \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

(ii) Firstly, we recognise that terminal velocity is achieved when $\ddot{x} = 0 \Rightarrow g - kv^2 = 0$

$$\Rightarrow V_T = \sqrt{\frac{g}{k}} \quad \& \text{ so } u = \sqrt{\frac{g}{k}}$$

ALTERNATIVE SOLUTIONS AVAILABLE - SEE SUPPLEMENT

... Continued Over

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments
Q 15(d) (ii) Cont	
The appropriate acceleration equation is $\frac{dv}{dt} = g - kv^2$	
$\Rightarrow \quad \frac{dv}{dt} = k\left(\frac{g}{k} - v^2\right)$ $= k(u^2 - v^2)$	$u = \sqrt{\frac{g}{k}}$
$\Rightarrow \int \frac{dv}{u^2 - v^2} = k\int dt$	Variables Separable Approach
$\Rightarrow \frac{1}{2u}\ln\left\|\frac{u+v}{u-v}\right\| = kt + C_1$ $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| + c$	
When $t = 0,\ v = 0 \Rightarrow C_1 = 0$. Standard Partial Fractions Result	
So $kt = \frac{1}{2u}\ln\left\|\frac{u+v}{u-v}\right\|$	✓✓✓ correct solution
By data, $t = T$ when $v = \frac{u}{2}$ & so we have :- $kT = \frac{1}{2u}\ln\left\|\frac{u + u/2}{u - u/2}\right\|$	✓✓ Substantial progress
$\Rightarrow kT = \frac{1}{2u}\ln(3)$ upon simplifying	✓ Basic progress
$\Rightarrow T = \frac{1}{2uk}\ln(3)$ & $u^2 = \frac{g}{k}$ or equivalently $\frac{1}{k} = \frac{u^2}{g}$	
& consequently, $T = \frac{1}{2u} \times \frac{u^2}{g}\ln(3)$, that is,	ALTERNATIVE SOLUTIONS AVAILABLE - SEE SUPPLEMENT
$T = \frac{u}{2g}\ln(3)$, as required.	

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments
Question 16 (a)(i) Consider the horizontal direction firstly. $\ddot{x} = -0.1\dot{x}$ or equivalently $\frac{dv}{dt} = -0.1v$ $\Rightarrow \int \frac{dv}{v} = -0.1\int dt$ $\Rightarrow \ln(v) = -0.1t + C$ & $\begin{cases} v = u \text{ when} \\ t = 0 \\ \Rightarrow C = \ln(u) \end{cases}$ $\therefore \ln\left(\frac{v}{u}\right) = -0.1t$ $\Rightarrow v = ue^{-0.1t}$ So $\int \frac{dx}{dt} = \int ue^{-0.1t}dt \Rightarrow x = -10ue^{-0.1t} + C_1$ & $\begin{cases} x = 0 \text{ when} \\ t = 0 \\ \Rightarrow C_1 = 10u \end{cases}$ $\therefore x = 10u - 10ue^{-0.1t}$ OR $x(t) = 10u\left(1 - e^{-0.1t}\right)$, as required.	Variables Separable Approach ✓✓ process ✓ progress
Now consider the vertical direction. $\ddot{y} = -g - 0.1\dot{y} \Rightarrow \frac{dv}{dt} = -g - 0.1v$ $\Rightarrow \int \frac{dv}{g + 0.1v} = -\int dt$ $\Rightarrow 10\int \frac{0.1\,dv}{g + 0.1v} = -\int dt$ $\Rightarrow 10\ln(g + 0.1v) = -t + C_2$ & $\begin{cases} v = 0 \text{ when} \\ t = 0 \\ \Rightarrow C_2 = 10\ln(g) \end{cases}$ $\therefore 10\ln(g + 0.1v) = -t + 10\ln(g)$ ... / Continued Over	Variables Separable Approach

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments
Q16 Cont (a)(i) Cont	
$10\ln(g+0.1v) = -t + 10\ln(g)$	
$\Rightarrow 10\ln(g+0.1v) - 10\ln(g) = -t$	✓✓✓ correct solution for $y(t)$;
$\Rightarrow 10\ln\left(\frac{g+0.1v}{g}\right) = -t$	✓✓ for substantial progress;
$\Rightarrow \ln\left(\frac{g+0.1v}{g}\right) = -0.1t$	✓ for basic progress
$\Rightarrow \frac{g+0.1v}{g} = e^{-0.1t}$	ALTERNATIVE SOLUTIONS AVAILABLE
$\Rightarrow 0.1v = g\left(e^{-0.1t}-1\right)$	
OR	
$v = 10g\left(e^{-0.1t}-1\right)$ or equivalently $\frac{dy}{dt} = 10g\left(e^{-0.1t}-1\right)$	
Therefore $\int \frac{dy}{dt}\,dt = \int 10g\left(e^{-0.1t}-1\right)dt$	
$\Rightarrow y = 10g\left(-10e^{-0.1t}-t\right)+c_3$ & $\left\{ y=0 \text{ when } t=0 \Rightarrow c_3 = 100g \right.$	
So $y = 10g\left(-10e^{-0.1t}-t\right)+100g$	
i.e. $y(t) = 100g\left(1-e^{-0.1t}\right)-10gt$, as required.	

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments

Q16 Cont (a) (ii)

We have $\dot{x}(t) = u e^{-0.1t}$ ... (I) ; $\dot{y}(t) = 100\left(e^{-0.1t} - 1\right)$ ... (III)

$x(t) = 10u\left(1 - e^{-0.1t}\right)$ ... (II) $y(t) = 1000\left(1 - e^{-0.1t}\right) - 100t$ ... (IV)

VERSION 1: Package hits the ground when $x = 4000$

$$\Rightarrow \quad 4000 = 10u\left(1 - e^{-0.1t}\right)$$

$$\Rightarrow \quad \frac{400}{u} = 1 - e^{-0.1t}$$

$$\Rightarrow \quad e^{-0.1t} = \frac{u - 400}{u} \quad \text{... (V)}$$

At impact $\tan(45^\circ) = \left|\frac{\dot{y}}{\dot{x}}\right|$ or equivalently $\tan(-45^\circ) = \frac{\dot{y}}{\dot{x}}$

$$\Rightarrow \quad 1 = \frac{100\left(1 - e^{-0.1t}\right)}{u e^{-0.1t}} \quad \text{or} \quad -1 = \frac{100\left(e^{-0.1t} - 1\right)}{u e^{-0.1t}} \quad \text{(VI)}$$

Substituting (V) into (VI) :

$$1 = \frac{100\left(1 - \frac{u-400}{u}\right)}{u\left(\frac{u-400}{u}\right)} \Rightarrow 1 = \frac{100 \times 400/u}{u(u-400)/u}$$

$$\Rightarrow \quad u(u-400) = 40000$$

$$\Rightarrow \quad u^2 - 400u - 40\,000 = 0$$

$$\Rightarrow \quad u = \left\{400 \pm \sqrt{(400)^2 + 4(40000)}\right\}/2$$

$$= \left\{400 \pm \sqrt{2 \times (400)^2}\right\}/2$$

i.e. $u = 200 \pm 200\sqrt{2}$. However, $u > 0$

So $u = 200 + 200\sqrt{2}$ only. Substituting this equation into (V) gives $e^{-0.1t} = \frac{200 + 200\sqrt{2} - 400}{200 + 200\sqrt{2}} = \frac{200(\sqrt{2}-1)}{200(\sqrt{2}+1)}$ or $\frac{\sqrt{2}-1}{\sqrt{2}+1}$

So $e^{0.1t} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$ or $3 + 2\sqrt{2}$

$$\Rightarrow \quad t = 10\ln\left(3 + 2\sqrt{2}\right) \approx 17.6\,s \quad (17.627471\ldots)$$

Comments

$g = 10\,m/s^2$

✓✓✓ for correct solution

IMPACT AT 45° to the HORIZONTAL

(At impact)

✓✓ for minor errors/ omissions

Quadratic Formula

✓ basic progress

Rationalising denominator

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

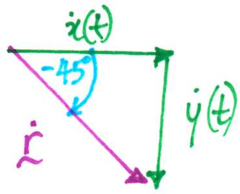
Suggested Solution (s)	Comments

Question 16 (a) (ii) VERSION 2: As before we have:

$$\begin{cases} \dot{x}(t) = ue^{-0.1t} \\ x(t) = 10u\left(1-e^{-0.1t}\right) \end{cases} \quad ; \quad \begin{cases} \dot{y}(t) = 100\left(e^{-0.1t}-1\right) \\ y(t) = 1000\left(1-e^{-0.1t}\right) - 100t \end{cases}$$

& $4000 = 10u\left(1-e^{-0.1t}\right)$ ($x(t) = 4000$)

Also, $\dot{x}(t) = -\dot{y}(t)$ at impact (Angle relative to horizontal is 45°)



$$\therefore \quad 1 = \frac{100\left(1-e^{-0.1t}\right)}{ue^{-0.1t}}$$

$$\Rightarrow \quad ue^{-0.1t} = 100\left(1-e^{-0.1t}\right) \quad \cdots (\alpha)$$

From $4000 = 10u\left(1-e^{-0.1t}\right) \Rightarrow u = \dfrac{400}{1-e^{-0.1t}} \cdots (\beta)$

Substituting (β) into (α) gives $\dfrac{400}{1-e^{-0.1t}} \cdot e^{-0.1t} = 100\left(1-e^{-0.1t}\right)$

$$\Leftrightarrow \quad \frac{4e^{-0.1t}}{1-e^{-0.1t}} = 1-e^{-0.1t}$$

$$\Leftrightarrow \quad \frac{4}{e^{0.1t}-1} = \frac{e^{0.1t}-1}{e^{0.1t}}$$

$$\Leftrightarrow \quad 4e^{0.1t} = \left(e^{0.1t}-1\right)^2$$

$$\Leftrightarrow \quad \left(e^{0.1t}\right)^2 - 6e^{0.1t} + 1 = 0$$

$$\Leftrightarrow \quad e^{0.1t} = \frac{6 \pm \sqrt{32}}{2} \text{ or } 3 \pm 2\sqrt{2}$$

$\Leftrightarrow \quad t = 10\ln\left(3 \pm 2\sqrt{2}\right)$ However $t > 0$ (physical constraint)

& so we reject $t = 10\ln\left(3-2\sqrt{2}\right)$.

Hence $t = 10\ln\left(3+2\sqrt{2}\right)$

$\Rightarrow$ $t \approx 17.6$ s as before ($t = 17.627471\ldots$)

Comments:

✓✓✓ for correct solution

✓✓ for minor errors/omissions

✓ for basic progress

Quadratic Formula

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments
Question 16(b) $a^2+b^2=c^2$ (Pythagoras' Theorem) $\Rightarrow (a^2+b^2)c = c^3$ or $a^2c+b^2c=c^3$ $\because a<c$ & $b<c$ $\therefore a^3<a^2c$ & $b^3<b^2c$ ← (Adding Inequalities) Consequently $a^3+b^3<a^2c+b^2c$ or equivalently $a^3+b^3<c^3$, as required. ALTERNATIVE SOLUTIONS: VERSION 1 We have $a^2+b^2=c^2$ (Pythagoras' Theorem) Let us assume $a^3+b^3<c^3$ or equivalently $a^3+b^3<(a^2+b^2)^{3/2}$ $\iff (a^3+b^3)^2<(a^2+b^2)^3$ $\iff a^6+2a^3b^3+b^6<a^6+3a^4b^2+3a^2b^4+b^6$ $\iff 2a^3b^3<3a^4b^2+3a^2b^4$ $\iff 2ab<3a^2+3b^2$; $ab\neq 0$ $\iff 0<2a^2+2b^2+(a-b)^2$ which is true as $x^2>0$ $\forall x\in\mathbb{R}\setminus\{0\}$ and certainly for $x>0$. VERSION 2: We note that $a^4b^2+a^2b^4=(a^2b)^2+(ab^2)^2-2a^2b\cdot ab^2+2a^2b\cdot ab^2$ $=(a^2b-ab^2)^2+2a^3b^3$ i.e. $a^4b^2+a^2b^4\geq 2a^3b^3$ as $(a^2b-ab^2)^2\geq 0$ $\forall a,b\in\mathbb{R}$ ...(I) Also $(a^2+b^2)^3=a^6+3a^4b^2+3a^2b^4+b^6$ $>a^6+a^4b^2+a^2b^4+b^6$ $\geq a^6+2a^3b^3+b^6$ using (I) $=(a^3+b^3)^2$ So $(a^2+b^2)^3>(a^3+b^3)^2$. However, $a^2+b^2=c^2$ $\therefore (c^2)^3>(a^3+b^3)^2$ $\Rightarrow (c^3)^2>(a^3+b^3)^2\,(>0)$ $\iff c^3>a^3+b^3$, as required.	✓✓ for argument

2021 Year 12 Mathematics Extension 2 Task 4 SOLUTIONS

Suggested Solution (s)	Comments
Question 16 (c) (i), For $I_{2n+1} = \int_0^1 x^{2n+1} \cdot e^{x^2} dx$, let $\begin{cases} u = x^{2n} \Rightarrow du = 2n x^{2n-1} dx \\ dv = x e^{x^2} dx \Rightarrow v = \frac{1}{2} e^{x^2} \end{cases}$ Then $I_{2n+1} = \frac{1}{2} x^{2n} \cdot e^{x^2} \Big\|_0^1 - n \int_0^1 x^{2n-1} \cdot e^{x^2} dx$ $= \frac{e}{2} - n \cdot I_{2n-1}$, as required	✓✓ Correct Solution ✓ progress in attempting to use integration by parts
(ii) $2\int_0^1 x^{2n-1} (1+x^2) e^{x^2} dx = 2\int_0^1 \left(x^{2n-1} \cdot e^{x^2} + x^{2n+1} \cdot e^{x^2}\right) dx$ $= 2\left(I_{2n-1} + I_{2n+1}\right)$ $= 2\left(I_{2n-1} + \frac{e}{2} - n I_{2n-1}\right)$ using Reduction Formula from (i)	1 mark
For $f(x) = x^{2n-1} \cdot e^{x^2}$; f is increasing on $\mathbb{R}$ and, in particular, on $x \in [0,1]$. Justification: $f'(x) = e^{x^2} \cdot (2n-1) x^{2n-2} + 2x \cdot e^{x^2} \cdot x^{2n-1}$ $= e^{x^2} \left((2n-1) \cdot x^{2n-2} + 2x^{2n}\right)$ $= x^{2n-2} \cdot e^{x^2} \cdot \left(x^2 + 2n - 1\right)$ $\geq e^{x^2} \cdot (x^2+1)$ as $n \in \mathbb{N}$ > 0 Also $f(0) = 0$; $f(1) = e \Rightarrow I_{2n-1} > 0$ (properties of integrals) Hence $n I_{2n-1} \geq I_{2n-1}$; $n \in \mathbb{N}$. $\therefore 2\left(I_{2n-1} + \frac{e}{2} - n I_{2n-1}\right) \leq 2\left(n I_{2n-1} + \frac{e}{2} - n I_{2n-1}\right)$ $= 2 \cdot \frac{e}{2}$ or e. Hence, $2\int_0^1 x^{2n-1} (1+x^2) e^{x^2} dx \leq e$, as required.	+ 2 marks for coherent argument ALTERNATIVE SOLUTIONS AVAILABLE

Question 16 (c) (ii)

OTHERWISE APPROACH: On $[0,1]$, $x^{2n-1}(1+x^2)e^{x^2} \geq 0$ with greatest lower bound $= 0$ & least upper bound $= 1$. The lub or supremum is achieved when $x=1$.

So $2\int_0^1 x^{2n-1}(1+x^2)\cdot e^{x^2}\,dx > 0$, noting glb or infimum achieved when $x=0$.

Also,
$$2\int_0^1 x^{2n-1}(1+x^2)\cdot e^{x^2}\,dx \leq 2\int_0^1 x^{2n-1}(1+1^2)\cdot e^{1^2}\,dx$$
$$= 4e\int_0^1 x^{2n-1}\,dx$$
$$= 4e\left[\frac{x^{2n}}{2n}\right]_0^1$$
$$= \frac{2e}{n}$$
$$\leq e \quad \text{when } \frac{2}{n} \leq 1 \text{, i.e., } n \geq 2 \;\&\; n \in \mathbb{N}.$$

It only remains to show $2\int_0^1 x(1+x^2)e^{x^2}\,dx \leq e$ which is equivalent to showing $\int_0^1 2xe^{x^2}\,dx + \int_0^1 x^2\cdot(2xe^{x^2})\,dx \leq e$.

Now $\int_0^1 2xe^{x^2}\,dx = e^{x^2}\Big|_0^1 = e-1$ & $\int_0^1 \underbrace{x^2}_{u}\overbrace{(2xe^{x^2})\,dx}^{dv} = x^2e^{x^2}\Big]_0^1 - \int_0^1 2xe^{x^2}\,dx = e-(e-1)$.

Hence,
$$2\int_0^1 x(1+x^2)e^{x^2}\,dx = (e-1) + e - (e-1)$$
$$= e$$

Thus,
$$2\int_0^1 x^{2n-1}(1+x^2)\cdot e^{x^2}\,dx \leq e \quad \forall\, n \in \mathbb{N}.$$

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Question 11 (d) (ii)

Sample Answer:

$z^7 - 1 = 0$

$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \quad (1)$

$\dfrac{z^6 + z^5 + z^4 + z^3 + z^2 + z + 1}{z^3} = \dfrac{0}{z^3}$

$z^3 + z^2 + z + 1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dfrac{1}{z^3} = 0$

$\left(z^3 + \dfrac{1}{z^3}\right) + \left(z^2 + \dfrac{1}{z^2}\right) + \left(z + \dfrac{1}{z}\right) + 1 = 0$

since $z^3 + \dfrac{1}{z^3} = 2\cos 3\theta$, $z^2 + \dfrac{1}{z^2} = 2\cos 2\theta$ and $z + \dfrac{1}{z} = 2\cos\theta$

$\Rightarrow \; 2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0 \quad (2)$

The solutions of (1) are the non-real seventh roots of unity, so $\theta = \pm\dfrac{2\pi}{7}, \pm\dfrac{4\pi}{7}, \pm\dfrac{6\pi}{7}$

$\therefore$ Since (1) and (2) are equivalent then $\dfrac{2\pi}{7}$, $\dfrac{4\pi}{7}$ and $\dfrac{6\pi}{7}$ are also solutions to (2).

Let $\theta = \dfrac{2\pi}{7}$ in (ii):

$$\therefore \; 2\cos 3\left(\frac{2\pi}{7}\right) + 2\cos 2\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{2\pi}{7}\right) + 1 = 0$$

$$2\left(\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right) = -1$$

$$\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

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15 (d) (ii) (3 marks)
Outcomes Assessed: M1.3/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Finds $t=\dfrac{1}{2\sqrt{kg}}\ln\left\|\dfrac{\sqrt{g}+\sqrt{kv}}{\sqrt{g}-\sqrt{kv}}\right\|$	2
• Correctly separates the variables OR • Finds $\dfrac{dt}{dv}=\dfrac{1}{g-kv^2}$	1

Sample Answer:
Relation between v and t:

$$\frac{dv}{dt}=g-kv^2$$

$$\sqrt{k}dt=\frac{\sqrt{k}dv}{g-\left(\sqrt{kv}\right)^2}$$

$$\sqrt{k}dt=\frac{\sqrt{k}dv}{g-kv^2}$$

$$\sqrt{k}dt=\left\{\frac{1}{\sqrt{g}-\sqrt{kv}}+\frac{1}{\sqrt{g}+\sqrt{kv}}\right\}\frac{\sqrt{k}dv}{2\sqrt{g}}$$

$$2\sqrt{kg}t+C=\ln\left|\frac{\sqrt{g}+\sqrt{kv}}{\sqrt{g}-\sqrt{kv}}\right|,$$

$t=0, v=0 \Rightarrow C=0,$

$$t=\frac{1}{2\sqrt{kg}}\ln\left|\frac{\sqrt{g}+\sqrt{kv}}{\sqrt{g}-\sqrt{kv}}\right| \qquad (1)$$

If $v=\dfrac{U}{2}$ and $k=\dfrac{g}{U^2}$, then from (1) $t=\dfrac{U}{2g}\ln 3.$

$$\frac{1}{g-kv^2}=\frac{1}{\left(\sqrt{g}-\sqrt{kv}\right)\left(\sqrt{g}+\sqrt{kv}\right)}$$

$$\frac{1}{\left(\sqrt{g}-\sqrt{kv}\right)\left(\sqrt{g}+\sqrt{kv}\right)}=\frac{A}{\left(\sqrt{g}-\sqrt{kv}\right)}+\frac{B}{\left(\sqrt{g}+\sqrt{kv}\right)}$$

$$v=A\left(\sqrt{g}+\sqrt{kv}\right)+B\left(\sqrt{g}-\sqrt{kv}\right)$$

$$v=\frac{\sqrt{g}}{\sqrt{k}},\quad 1=2A\sqrt{g}\quad\Rightarrow A=\frac{1}{2\sqrt{g}}$$

$$v=-\frac{\sqrt{g}}{\sqrt{k}},\quad 1=2B\sqrt{g}\quad\Rightarrow B=\frac{1}{2\sqrt{g}}$$

$$\frac{v}{g-kv^2}=\frac{1}{2\sqrt{g}\left(\sqrt{g}-\sqrt{kv}\right)}+\frac{1}{2\sqrt{g}\left(\sqrt{g}+\sqrt{kv}\right)}$$

Supplement – Alternative Solutions
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Alternatively:

$$\frac{dv}{dt} = g - kv^2$$

$$\frac{dt}{dv} = \frac{1}{g - kv^2}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$t = \frac{1}{2\sqrt{g}} \int_0^{\frac{U}{2}} \left(\frac{1}{\sqrt{g} - \sqrt{kv}} + \frac{1}{\sqrt{g} + \sqrt{kv}} \right) dv$$

$$= \frac{1}{2\sqrt{kg}} \left[-\ln\left|\sqrt{g} - \sqrt{k}v\right| + \ln\left|\sqrt{g} + \sqrt{k}v\right| \right]$$

$$= \frac{1}{2\sqrt{kg}} \left[\ln\left| \frac{\sqrt{g} + \sqrt{k}v}{\sqrt{g} - \sqrt{k}v} \right| \right]_v^{\frac{U}{2}}$$

$$= \frac{1}{2\sqrt{kg}} \left(\ln\left| \frac{\sqrt{g} + \sqrt{k} \times \frac{U}{2}}{\sqrt{g} - \sqrt{k} \times \frac{U}{2}} \right| - \ln\left| \frac{\sqrt{g}}{\sqrt{g}} \right| \right)$$

$$= \frac{1}{2\sqrt{kg}} \ln\left| \frac{2\sqrt{g} + U\sqrt{k}}{2\sqrt{g} - U\sqrt{k}} \right|$$

$$\frac{1}{g - kv^2} = \frac{1}{\left(\sqrt{g} - \sqrt{k}v\right)\left(\sqrt{g} + \sqrt{k}v\right)}$$

$$\frac{1}{\left(\sqrt{g} - \sqrt{k}v\right)\left(\sqrt{g} + \sqrt{k}v\right)} = \frac{A}{\left(\sqrt{g} - \sqrt{k}v\right)} + \frac{B}{\left(\sqrt{g} + \sqrt{k}v\right)}$$

$$v = A\left(\sqrt{g} + \sqrt{k}v\right) + B\left(\sqrt{g} - \sqrt{k}v\right)$$

$$v = \frac{\sqrt{g}}{\sqrt{k}}, \quad 1 = 2A\sqrt{g} \quad \Rightarrow A = \frac{1}{2\sqrt{g}}$$

$$v = -\frac{\sqrt{g}}{\sqrt{k}}, \quad 1 = 2B\sqrt{g} \quad \Rightarrow B = \frac{1}{2\sqrt{g}}$$

$$\frac{v}{g - kv^2} = \frac{1}{2\sqrt{g}\left(\sqrt{g} - \sqrt{k}v\right)} + \frac{1}{2\sqrt{g}\left(\sqrt{g} + \sqrt{k}v\right)}$$

At terminal velocity:

$$0 = g - kU^2 \Rightarrow \sqrt{k} = \frac{\sqrt{g}}{U}$$

$$\therefore t = \frac{1}{2\sqrt{g}\left(\frac{\sqrt{g}}{U}\right)} \ln\left| \frac{2\sqrt{g} + U\frac{\sqrt{g}}{U}}{2\sqrt{g} - U\frac{\sqrt{g}}{U}} \right|$$

$$= \frac{U}{2g} \ln \frac{3\sqrt{g}}{\sqrt{g}}$$

$$= \frac{U}{2g} \ln 3.$$